Defining positive definite arithmetical functions and a partial order on the set of arithmetical functions by using matrix inequalities

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Abstract

In this presentation we make use of the Löwner order on square matrices and induce a partial order on the set

$$\mathcal{A} = \{f : \mathbb{Z}^+ \to \mathbb{R}\}$$

of real-valued arithmetical functions. If f and g are given arithmetical functions, we define that $f \preceq g$ if and only if $(S)_f \preceq (S)_g$ for all $S = \{x_1, x_2, \ldots, x_n\} \subset \mathbb{Z}^+$ and all $n = 1, 2, \ldots$, where $(S)_f = [f(\gcd(x_i, x_j))]$ and $(S)_g = [g(\gcd(x_i, x_j))]$ are the GCD matrices of the set S with respect to function f and g, respectively.

Positive definiteness of a function $f : \mathbb{R} \to \mathbb{C}$ is usually defined by demanding that the matrix $[f(x_i - x_j)] \in M_n$ is positive semidefinite for all choices of points $\{x_1, x_2, \ldots, x_n\} \subset \mathbb{R}$ and all $n = 1, 2, \ldots$ [1, p. 400]. However, this definition does not work for arithmetical functions defined only on positive integers. By using our newly defined partial order it is natural to define that an arithmetical function f is positive definite if and only if $f \succeq \mathbf{0}$, where $\mathbf{0}$ is the constant function having all of its values equal to 0.

We shall study the basic properties of our partial order $\preceq$ on $\mathcal{A}$ as well as properties of positive definite arithmetical functions. We also consider some elementary examples.

Keywords

Arithmetical function, Positive definite function, Partial order, Löwner order, GCD matrix.

References

[1] R. A. Horn and C. R. Johnson (1985). *Matrix Analysis*. New York: Campridge University Press.